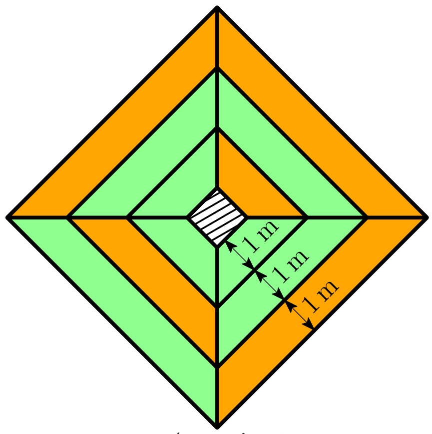
16. January 2026

To gain full marks you need to justify all the answers, not only to calculate them.

Problem 1.
There is a square platform with a side length of 1 meter (dashed square in the middle) and there are stairs leading down from each side of the platform. Each stair is 1 meter wide. Michal and Vítek coloured the top side of each stair. Michal used the colour green and Vítek used the colour orange. You can see the top view of the final result in the picture. Who used more colour?



(Mária Dományová)

Solution

As we can see, the whole picture is based on squares. Indeed, the border of outside stairs is a square with side length of 7 meters. Then, the border of the second the biggest stair is square with side of length 5 meters, then 3 meters and the center i dashed square of side length 1 meter.

By symmetry of the picture, we can see that:

$$Area_{orange} = \frac{3}{4}\cdot(7^2-5^2)+\frac{1}{4}\cdot(5^2-3^2)+\frac{1}{4}\cdot(3^2-1^1) = 18+4+2 = 24\ m^2,$$
$$Area_{green} = \frac{1}{4}\cdot(7^2-5^2)+\frac{3}{4}\cdot(5^2-3^2)+\frac{3}{4}\cdot(3^2-1^1) = 6+12+6 = 24\ m^2.$$

We can see that they both used the same amount of colour.

Problem 2.
Let ABC be an equilateral triangle. Points D, E, F, and G lie inside the sides of the triangle such that BD is perpendicular to AC, DE is perpendicular to BC, EF is perpendicular to BD, and FG is perpendicular to BC. Determine the ratio $|BG| : |GE| : |EC|$.

(Marián Macko)

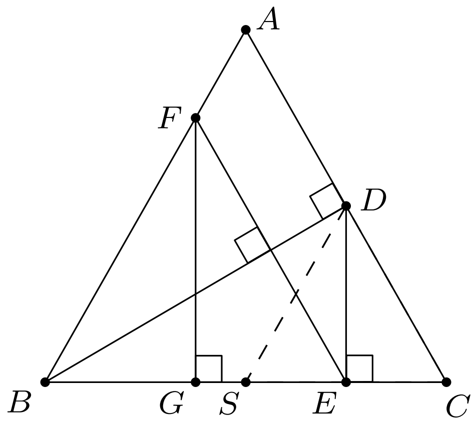
Solution

Let's consider point S to be the midpoint of BC. Point D is the midpoint of AC as an altitude to a side of an equilateral triangle is the same as its perpendicular bisector. Since DS is the midsegment of triangle, ABC triangle SCD is equilateral. Now point E would be the midpoint of SC, so $|EC| = \frac{1}{4}|BC|$.

Now let's look at triangle BEF. It is also an equilateral triangle (as EF is parallel to CA given us by orthogonality of EF and BC) with side length $|BE| = \frac{3}{4}|BC|$ and G is the midpoint of BE. So we can calculate that $|BG| = |GE| = \frac{3}{8}|BC|$.

The resulting ratio would be $|BG| : |GE| : |EC| = \frac{3}{8} : \frac{3}{8} : \frac{1}{4}$, which can be rewritten as

$$|BG| : |GE| : |EC| = 3 : 3 : 2.$$



Problem 3.
Let $ABCD$ be a trapezium satisfying $AB \parallel CD$ and $AB \perp AD$. There is a point X on side AD satisfying $|AX| : |XD| = 2 : 1$ and $|\sphericalangle CXD| = |\sphericalangle AXB|$. Assume the area of triangle BCX is equal to $16\ \text{cm}^2$. Calculate the area of the trapezium $ABCD$.

(Mária Dományová)

Solution

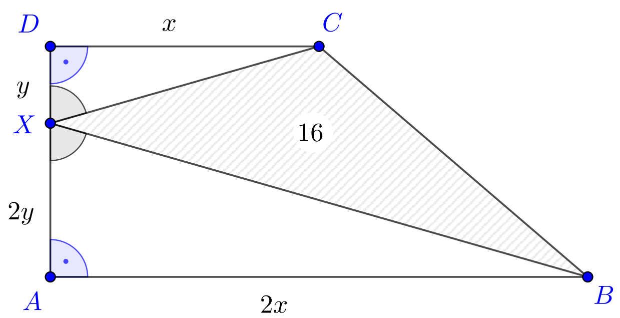
We are given that $AB \parallel CD$ and $AB \perp AD$. This means the side AD is perpendicular to both bases, so $\angle XAB = 90°$ and $\angle XDC = 90°$. Additionally, we are given that $|\angle AXB| = |\angle CXD|$. Since these two triangles share a right angle and a second equal angle, they are similar by the Angle-Angle (AA) criterion:

$$\triangle AXB \sim \triangle DXC$$

Due to this similarity, the ratio of corresponding sides is constant:

$$\frac{|AB|}{|CD|} = \frac{|AX|}{|XD|} = \frac{2}{1}$$

Substituting $|CD| = x$, we find that $|AB| = 2x$.



We can now calculate the area of the trapezium $ABCD$ in two different ways. First, we calculate the area as the sum of the three individual triangles $\triangle AXB$, $\triangle CXD$, and $\triangle BCX$:

$$\text{Area}(\triangle AXB) = \frac{1}{2} \cdot |AB| \cdot |AX| = \frac{1}{2}(2x)(2y) = 2xy$$
$$\text{Area}(\triangle CXD) = \frac{1}{2} \cdot |CD| \cdot |XD| = \frac{1}{2}(x)(y) = 0.5xy$$

Given that the area of $\triangle BCX$ is $16\ \text{cm}^2$, the total area is:

$$S_{total} = 2xy + 0.5xy + 16 = 2.5xy + 16 \tag{1}$$

Second, we calculate the total area using the standard formula for a trapezium, $S = \frac{a+c}{2} \cdot h$:

$$S_{total} = \frac{|AB| + |CD|}{2} \cdot |AD| = \frac{2x + x}{2} \cdot 3y = \frac{3x}{2} \cdot 3y = 4.5xy \tag{2}$$

By equating the two expressions (1) and (2) for the total area, we get:

$$4.5xy = 2.5xy + 16$$

Subtracting $2.5xy$ from both sides yields:

$$2xy = 16 \implies xy = 8$$

Finally, we substitute $xy = 8$ back into the trapezium area formula:

$$S_{total} = 4.5(8) = 36$$

Thus, the area of the trapezium $ABCD$ is $\mathbf{36\ cm^2}$.

Problem 4.

Let ABC be an acute scalene triangle. Denote by D and E the feet of the perpendiculars from A to BC and B to AC, respectively. Let X, Y be points such that $DXEY$ is a rhombus and X lies on segment AB. Assume that C lies inside $DXEY$ and that $|\sphericalangle CAY| = |\sphericalangle AYE|$. Find the measure of the angle $\sphericalangle ABE$.

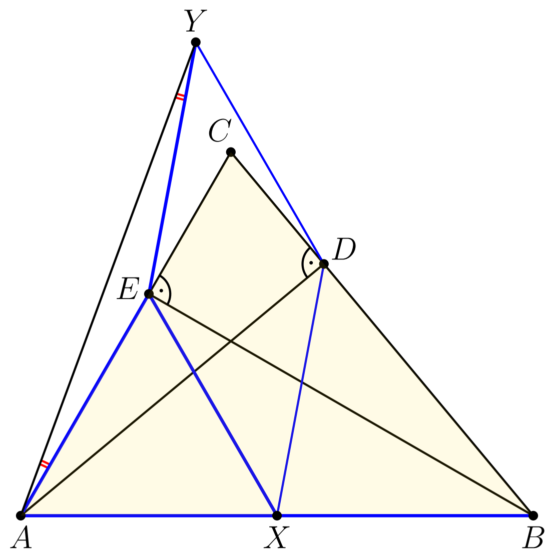
(Patrik Vrba)

Solution

From $|\sphericalangle CAY| = |\sphericalangle AYE|$ we know that AEY is an isosceles triangle, so $|EA| = |EY|$. Since $DXEY$ is a rhombus we get $|EA| = |EX|$.

As D and E are feet of perpendiculars, it follows that $|\sphericalangle ADB| = |\sphericalangle AEB| = 90°$. Therefore, from Thales's theorem, points D and E lie on a circle with diameter AB. Let us denote the center of this circle O. We know that O must lie on AB and also satisfy $|OD| = |OE|$, giving us $O = X$. From this we get $|AX| = |BX| = |DX| = |EX|$, making AXE an equilateral triangle and BDX an isosceles triangle.

Now we can calculate some angles. We have proven that AXE is equilateral, so $|\sphericalangle AXE| = 60°$ and therefore $|\sphericalangle BXE| = 120°$. As BXE is isosceles we finally conclude that $|\sphericalangle XBE| = |\sphericalangle ABE| = 30°$.



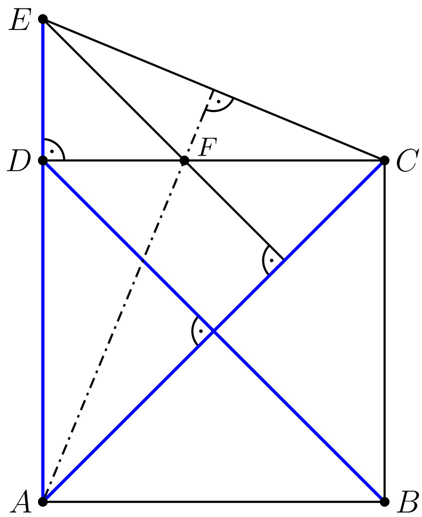
Problem 5.
Let $ABCD$ be a square. Let E be a point on the line AD such that $|AE| = |BD|$ and D lies between the points A and E. The perpendicular bisector of the segment CE intersects the line CD at a point F. Show that the line EF is parallel to the diagonal BD. Does this hold even if A lies between D and E?

(Marián Macko)

Solution

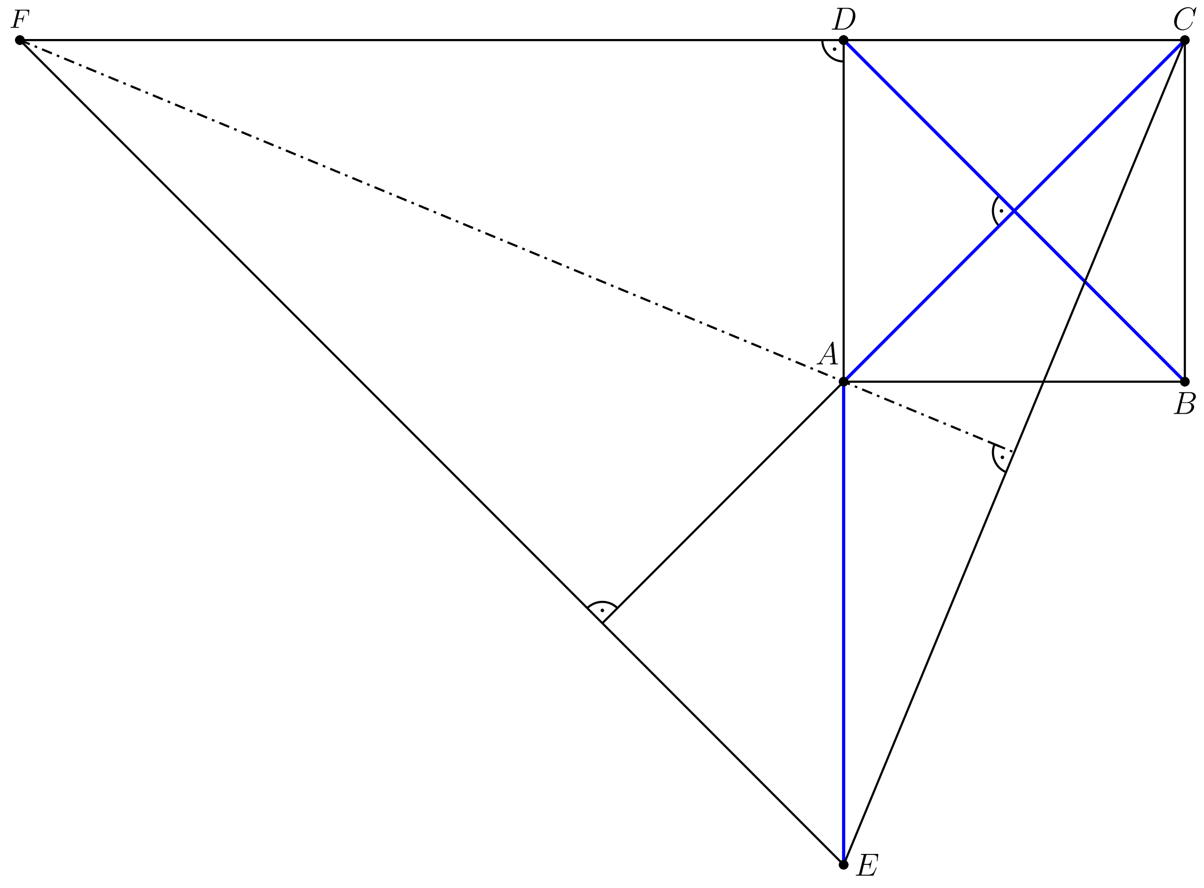
First we solve it when D lies between the points A and E. Because BD and AC are diagonals of a square, they have the same length. And hence we have $|AC| = |AE|$, from which we conclude that the triangle CAE is isosceles.

In isosceles triangle we know that the perpendicular bisector of its base coincides with the altitude. In this case the perpendicular bisector of CE is the altitude from A in ACE. Therefore we have that $AF \perp CE$. Because $ABCD$ is a square we also have that $CD \perp AD$ and thus CF is also the altitude in ACE. From that we conclude that F is the orthocenter of ACE. Because of that $EF \perp AC$. And because diagonals of square are perpendicular we also have $BD \perp AC$ and thus $BD \parallel EF$.



Now we look at what happens when A lies between D and E. We again have $|AC| = |AE|$ as in the first case. So again the triangle ACE is isosceles (in this case it will be an obtuse triangle).

From ACE being isosceles we can again conclude that the perpendicular bisector of CE is the altitude from A in ACE and CF is the altitude from C in ACE. Hence F is again the orthocenter of ACE. From that we again conclude that EF is the altitude in triangle AEC and hence $EF \perp AC$ and we still have that $BD \perp AC$, because diagonals are perpendicular. And thus again we have $BD \parallel EF$.

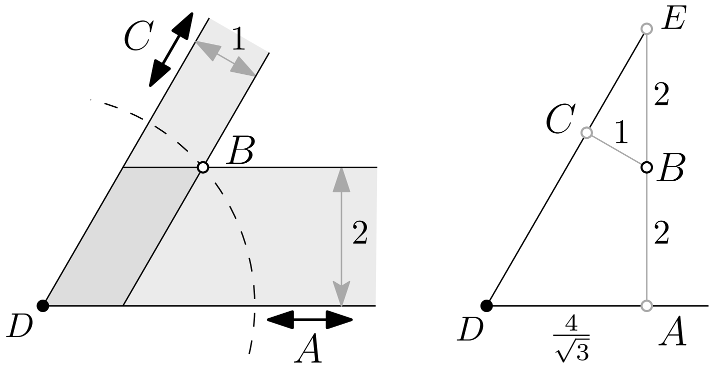

F
D
C
A
B
E

Problem 6.
Let $ABCD$ be a convex[1] quadrilateral with $|AB| = 2$, $|BC| = 1$ and $|\sphericalangle CDA| = 60°$. Find the maximum possible length of diagonal BD and justify why it cannot be larger.

(Josef Tkadlec)

Solution

The trick is to start drawing the figure from the right points. We begin with point D and draw rays DA (to the right) and DC (diagonally upwards). Where can point B lie? Due to convexity and the condition $|AB| = 2$, it can only be in a horizontal strip of height 2 "above" the ray DA. Similarly, due to the condition $|BC| = 1$, B must lie in a strip of width 1 "to the right" of the ray DC. Thus, it must lie at the intersection of these strips, which is a parallelogram. The furthest point from D is obviously the opposite vertex of this parallelogram. Points A, C are then the feet of the perpendiculars from B. So, in $ABCD$, the angles at A and C are right angles, and the rest is straightforward algebra with "halves" of equilateral triangles. For example, we can let $E = AB \cap CD$, then $|BE| = 2|BC| = 2$, so $|AD| = \frac{1}{\sqrt{3}}|AE| = \frac{4}{\sqrt{3}}$, and from the Pythagorean theorem for $\triangle DAB$ we have $|DB|^2 = 16/3 + 4 = 28/3$, so $|DB| = \frac{2}{3}\sqrt{21}$.



[1]A polygon is called convex if all its interior angles are less than 180°.